

Einstein-Proca Model, Micro Black Holes, and Naked Singularities

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Abstract

The Einstein-Proca equations, describing a spin-1 massive vector field in general relativity, are studied in the static, spherically-symmetric case. The Proca field equation is a highly nonlinear wave equation, but can be solved to good accuracy in perturbation theory, which should be very accurate for a wide range of mass scales. The resulting first order metric reduces to the Reissner-Nordstrom solution in the limit as the range parameter μ goes to zero. The additional terms in the g_{00} metric are positive, as in Reissner-Nordstrom, in agreement with previous numerical solutions, and hence involve naked singularities. Note: This paper was published in General Relativity and Gravitation, May 2002.

1 Introduction

An exact solution for the Einstein-Proca system for an idealized point particle has yet to be found [1], [2]. Such systems have been occasionally discussed in the literature, for example in Dereli et al. [3], and have been invoked by Tucker and Wang [4] in connection with dark matter gravitational interactions, where it was shown that such fields could explain in part the galactic rotation curves. Numerical solutions were found independently by Obukov and Vlachynsky [5] and Toussaint [6]. These latter two papers demonstrated the existence of naked singularities in this system. In this section, the system will be solved up to a final integral, which will then be subjected to perturbation analysis.

Consider a force modeled as a Proca interaction. During gravitational collapse, the equivalent of the force charge, referred to here as the Proca charge, would not be cancelled by an accumulation of opposite charges, as in electromagnetic interactions. The stress energy of the force field would therefore be expected to make contributions to the gravitational field of the spacetime surrounding the collapsed object. Because both the force and the associated gravitational field fall off exponentially, the effect on the spacetime surrounding a stellar-size black hole would be completely negligible.

On the other hand, it is thought that microscopic black holes may have been created in vast numbers during the Big Bang. These micro black holes would be expected to have a variety of different sizes, including, conceivably, some on the order of a femtometer across. For such objects, there is the possibility that associated fields of Proca-type would prevent the formation of event horizons, leaving a (short-lived) naked singularity. This, then, might be considered a counter-example to Penrose's cosmic censorship conjecture.

The equation for a particle exhibiting a spin-1 short or intermediate-range field in flat space is Proca's equation [7], which in the absence of currents is

$$\partial_a F^{ab} + \mu^2 A^b = 0 \quad (1)$$

where

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \quad (2)$$

The metric will be taken to have diagonal form $c^2, -1, -1, -1$. The quantity μ is a constant, interpreted as being proportional to the mass of the field quanta and inversely proportional to the range of the interaction.

Traditionally, the form of equation 1 was chosen for several good reasons. First and foremost, it gives an intuitively correct answer, which is a potential that rapidly falls off as r gets large. Second, it can be realized by adding a linear term to Maxwell's equations. Third, the equation is covariant, and finally, a Lagrangian exists, meaning this equation is extremal in a more general function space.

The Lagrangian density for the classic Proca system is:

$$\mathcal{L} = \sqrt{-g} (\alpha F_{ab} F^{ab} + \beta A_a A^a) \quad (3)$$

where g is the determinant of the metric, and α and β are constants. Varying this equation with respect to A^c returns equation 2, provided that $\beta/2\alpha = -\mu^2$. It turns out that the last term on the right in 3, which distinguishes the standard Proca from Maxwell, causes considerable difficulties in finding the solution to the general relativistic problem. These difficulties are absent in the Reissner-Nordstrom problem primarily due to the antisymmetry of F_{ab} . Nonetheless, considerable progress can be made, as will be demonstrated in the next section.

2 Derivation and Solution of the Field Equations

The metric for static spherical symmetry can be taken to have the form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (4)$$

Similar forms can also be written down for plane and hyperbolic symmetry: all subsequent steps in this paper could equally well be taken in those two cases. The Proca stress-energy tensor can be obtained from

$$T_{ab} = -\frac{\alpha_M}{8\pi} \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \sqrt{-g} \mathcal{L} \quad (5)$$

For a given field, the constant α_M is a parameter that tells how strongly the stress-energy of the field creates gravitation. This gravitational strength is so weak compared to the other forces that it is impractical to determine experimentally. Again for convenience, this constant and the factor of 8π shall be rolled into the constants α and β . Applying this formula to equation 3 results in

$$T_{ab} = 2\alpha F_a{}^d F_{bd} + \beta A_a A_b - \frac{1}{2} g_{ab} (\alpha F_{cd} F^{cd} + \beta A_c A^c) \quad (6)$$

The Proca stress energy, unlike the Maxwell stress-energy, is not traceless. Einstein's equations read

$$R_{ab} = \kappa \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \quad (7)$$

It is advantageous to recast the Proca equation in terms of ordinary partial derivatives:

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} F^{ab}) - \frac{\beta}{2\alpha} A^b = 0 \quad (8)$$

The Proca system corresponds to a choice of

$$\frac{\beta}{2\alpha} = -\mu^2 \quad (9)$$

We search for a solution of equations 3-8 where F_{ab} is of the form

$$F_{ab} = \begin{pmatrix} 0 & -A'_0 & 0 & 0 \\ A'_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

With these choices, the stress-energy tensor becomes

$$T_{ab} = \alpha A_0'^2 \begin{pmatrix} -e^{-\lambda} & 0 & 0 & 0 \\ 0 & e^{-\nu} & 0 & 0 \\ 0 & 0 & -r^2 e^{-\lambda-\nu} & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta e^{-\lambda-\nu} \end{pmatrix} +$$

$$+ \frac{\beta A_0^2}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\lambda-\nu} & 0 & 0 \\ 0 & 0 & r^2 e^{-\nu} & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta e^{-\nu} \end{pmatrix} \quad (11)$$

Einstein's equation then can be written down as

$$R_{00} = e^{\nu-\lambda} \left(\frac{\nu''}{2} - \frac{\nu' \lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu'}{r} \right) = -\kappa \alpha A_0'^2 e^{-\lambda} + \kappa \beta A_0^2 \quad (12)$$

$$R_{11} = \left(-\frac{\nu''}{2} + \frac{\nu' \lambda'}{4} - \frac{\nu'^2}{4} + \frac{\lambda'}{r} \right) = \kappa \alpha A_0'^2 e^{-\nu} \quad (13)$$

$$R_{22} = 1 + e^{-\lambda} \left(-1 - \frac{r \nu'}{2} + \frac{r \lambda'}{2} \right) = -\kappa \alpha r^2 A_0'^2 e^{-\lambda-\nu} \quad (14)$$

Of course, $R_{33} = R_{22} \sin^2 \theta$. Finally, the equation for the massive vector field is given by

$$A_0'' + \frac{2}{r} A_0' - \left(\frac{\lambda'}{2} + \frac{\nu'}{2} \right) A_0' + \frac{\beta}{2\alpha} e^{\lambda} A_0 = 0 \quad (15)$$

On the face of it, these equations are not dissimilar to Einstein-Maxwell, differing only by the inclusion of two rather innocuous terms. In fact, these small changes result in a tremendous complications, as will soon be seen. In the first place, unlike Einstein-Maxwell, the enormous simplification of $\lambda' + \nu' = 0$ does not occur. Indeed, multiplying equation 12 by $e^{-\nu+\lambda}$ and adding to equation 13 yields

$$\frac{\nu'}{r} + \frac{\lambda'}{r} = \kappa \beta A_0'^2 e^{\lambda-\nu} \quad (16)$$

Solving this equation for λ' and substituting into equation 14 results, after some algebra, in:

$$e^{\lambda} = \frac{1 + r \nu' - \kappa \alpha r^2 A_0'^2 e^{-\nu}}{1 + \frac{1}{2} \kappa \beta r^2 A_0'^2 e^{-\nu}} \quad (17)$$

So the function e^{λ} has been solved in terms of the other two functions. This result, when substituted into the 00 and 11 equations, makes them identical. Using the last two equations, the remaining equations for ν and A_0 can be written as:

$$\nu'' + \nu'^2 + \frac{2\nu'}{r} = -2\kappa \alpha A_0'^2 e^{-\nu} + \left(2 + \frac{r \nu'}{2} \right) \kappa \beta A_0'^2 e^{-\nu} \frac{1 + r \nu' - \kappa \alpha r^2 A_0'^2 e^{-\nu}}{1 + \frac{1}{2} \kappa \beta r^2 A_0'^2 e^{-\nu}} \quad (18)$$

$$A_0'' + \frac{2}{r} A_0' = \frac{\beta}{2\alpha} A_0 \left(-1 + \kappa r A_0 A_0' e^{-\nu} \right) \frac{1 + r \nu' - \kappa \alpha r^2 A_0'^2 e^{-\nu}}{1 + \frac{1}{2} \kappa \beta r^2 A_0'^2 e^{-\nu}} \quad (19)$$

The equation for ν can be significantly simplified by the substitution

$$e^{\nu} = f \quad (20)$$

where $f = f(r)$. Substituting this into equation 18 results in

$$f'' + \frac{2}{r} f' = -2\kappa \alpha A_0'^2 + \kappa \beta A_0^2 \left(2 + \frac{r f'}{2f} \right) \left[\frac{f + r f' - \kappa \alpha r^2 A_0'^2}{f + \frac{1}{2} \kappa \beta r^2 A_0'^2} \right] \quad (21)$$

Similarly, in equation 19:

$$A_0'' + \frac{2}{r}A_0' = \frac{\beta}{2\alpha}A_0 \left(-1 + \frac{\alpha\kappa r A_0 A_0'}{f} \right) \left[\frac{f + rf' - \kappa\alpha r^2 A_0'^2}{f + \frac{1}{2}\kappa\beta r^2 A_0'^2} \right] \quad (22)$$

It may be there is an exact solution for these two equations, however finding it would be a matter of experimentation and luck, given the cubic nonlinearities. A perturbative approach, on the other hand, has good chances of success, and can be quite accurate for reasonable values of the parameters of the theory. The procedure involves redefining all quantities so that they are dimensionless, using naturally-occurring parameters.

First, to get the Proca, it is necessary to define α and β . Let these be

$$\alpha = -\frac{1}{2}\epsilon_0 \quad (23)$$

and

$$\beta = \mu^2 \epsilon_0 \quad (24)$$

The quantity ϵ_0 fulfills the same function as the permittivity of free space in electromagnetism, but in this context pertains to the Proca interaction. μ is, of course, the standard range parameter. Next, set

$$x = \mu r \quad (25)$$

This redefines the r-coordinate in terms of a dimensionless parameter. The metric function f is already dimensionless; however A_0 has dimensions of Joules per Proca charge. Denote the Proca charge by q , in analogy with electromagnetism. Next, set

$$A = su \quad (26)$$

where

$$s = \epsilon_0^{-1} q \mu \quad (27)$$

The parameter s carries all the units of A . Substitute all these into the above equations and obtain the following two equations in terms of dimensionless variables only:

$$\left(u'' + \frac{2}{x}u' \right) \left(f + \frac{1}{2}\epsilon x^2 u^2 \right) f = u \left(f + \frac{1}{2}\epsilon u u' \right) \left(f + x f' + \frac{1}{2}\epsilon x^2 u'^2 \right) \quad (28)$$

$$\left(f'' + \frac{2}{x}f' - \epsilon u'^2 \right) \left(f + \frac{1}{2}\epsilon x^2 u^2 \right) f = \epsilon u^2 \left(2f + \frac{1}{2}x f' \right) \left(f + x f' + \frac{1}{2}\epsilon x^2 u'^2 \right) \quad (29)$$

where

$$\epsilon = \frac{\kappa q^2 \mu^2}{\epsilon_0} \quad (30)$$

is a small, dimensionless perturbation parameter, with $\kappa = G/c^4$. For a scale similar to that of the strong force, the factor μ^2 is quite large, $\approx 10^{30}$, and $\kappa \approx 10^{-44}$. The remaining term, q^2/ϵ_0 , is analogous to electromagnetic quantities, where the term would have magnitude of about 10^{-27} . Since the strong force is about 100 times stronger than the electromagnetic force, it follows that this combination of terms should be around 10^{-25} in the case under consideration. It appears therefore well justified

to consider ϵ a small quantity for a wide range of scale. The functions u and f may therefore be expanded:

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \quad (31)$$

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots \quad (32)$$

Inserting these expressions, the following zeroth order equations are obtained:

$$\left(f_0'' + \frac{2}{x}f_0'\right)f_0^2 = 0 \quad (33)$$

$$\left(u_0'' + \frac{2}{x}u_0'\right)f_0^2 = u_0 f_0 (f_0 + x f_0') \quad (34)$$

Equation 33 has the solution

$$f_0 = a + \frac{b}{x} \quad (35)$$

The second term on the right will be the usual Schwarzschild term, but will evidently be small, and more appropriately first order. Hence b will be taken to be zero, with $a = 1$, giving Minkowski space as the lowest order in the metric. With this choice, equation 34 has the usual flat space solution, which is

$$u_0 = c_0 \frac{e^{-x}}{x} + c_1 \frac{e^x}{x} \quad (36)$$

It is evident that $c_1 = 0$ in this case. The first order equations may be written:

$$\begin{aligned} &\left(u_1'' + \frac{2}{x}u_1'\right)f_0^2 + \left(u_0'' + \frac{2}{x}u_0'\right)f_0\left(2f_1 + \frac{1}{2}x^2u_0^2\right) = \\ &= u_0 f_0 \left(f_1 + x f_1' + \frac{1}{2}x^2 u_0'^2\right) + (f_0 + x f_0') \left(f_0 u_1 + u_0 f_1 + \frac{1}{2}u_0^2 u_0'\right) \end{aligned} \quad (37)$$

$$\begin{aligned} &\left(f_1'' + \frac{2}{x}f_1' - u_0'^2\right)f_0^2 + \left(f_0'' + \frac{2}{x}f_0'\right)\left(2f_0 f_1 + \frac{1}{2}x^2 u_0^2\right) = \\ &= u_0^2 \left(2f_0 + \frac{x}{2}f_0'\right)(f_0 + x f_0') \end{aligned} \quad (38)$$

The focus here is on equation 38, which yields the first-order correction to the metric. The homogeneous solution is again given by equation 35, except this time the constant solution will be discarded and the b/x term retained. This can be identified with the standard Schwarzschild term. In addition, a particular solution is needed. After substituting the functions f_0 and u_0 , the equation for f_1 becomes

$$f_1'' + \frac{2}{x}f_1' = c_0^2 \left(3 \frac{e^{-2x}}{x^2} + 2 \frac{e^{-2x}}{x^3} + \frac{e^{-2x}}{x^4}\right) \quad (39)$$

The particular solution of this equation is

$$f_{1p} = c_0^2 \left(\frac{1}{2} \frac{e^{-2x}}{x} + \frac{1}{2} \frac{e^{-2x}}{x^2} + \int \frac{e^{-2x}}{x} dx\right) \quad (40)$$

This expression is positive-definite, which will be important in the subsequent interpretation. The last term can be integrated by parts to give a slight simplification, which is

$$f_{1p} = \frac{c_0^2}{2} \left(\frac{e^{-2x}}{x^2} + \int_x^\infty \frac{e^{-2x}}{x^2} dx \right) \quad (41)$$

The metric function e^ν , with appropriate renormalization of the constants, can then be written in the form

$$e^\nu = 1 - \frac{2MG}{c^2 r} + \frac{q^2 G}{\epsilon_0 c^4} \left(\frac{e^{-2\mu r}}{r^2} + \mu^2 \int_r^\infty \frac{e^{-2\mu r}}{r^2} dr \right) \quad (42)$$

In the above equation, it has been assumed that the total classical energy of the field contributes to the gravitational field. In the limit as $\mu \rightarrow 0$, corresponding to an infinite range for the vector potential, a Reissner-Nordstrom spacetime is recovered.

3 Concluding Remarks

It is thought that numerous micro black holes may have been created in the early universe. Those black holes would be expected to evaporate over time due to emission of thermal radiation. The positive Proca terms in the above metric suggest the possibility that some of these objects might be devoid of event horizons, in agreement with the earlier numerical solutions of Obukov and Vlachynsky and Toussaint.

Another interesting property of the above solution is that the gravitational field is repulsive when the constants take on suitable values, because as r gets very small the exponential terms will dominate. One is left to speculate whether such repulsive effects could prevent complete catastrophic gravitational collapse.

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